it is seen that we have means at our disposal to separate any true periodicity of a variable from among its irregular changes, provided we can extend the time limits sufficiently.

The proof of this proposition lies outside the limits of this paper.

The application of the theory of probability to the investigation of what may be called "hidden" periodicities, an instance of which has here been given, may be further extended, and interesting results are obtained when a number of periodicities, such as those supposed to depend on the rotation of the sun about its axis, are critically examined. A full treatment of the subject will shortly appear in 'Terrestrial Magnetism.'

"The Vector Properties of Alternating Currents and other Periodic Quantities." By W. E. Sumpner, D.Sc. Communicated by O. Henrici, F.R.S. Received May 28,—Read June 17, 1897.

It has been well known for many years that the variations of a simple harmonic function, such as A cos pt, can be represented by the projection, on a fixed line, of a vector of constant length A supposed to revolve uniformly so as to complete one revolution in a time T given by the relation $p\mathrm{T} = 2\pi$. The angle between the revolving vector and the fixed line is pt at any instant t. In the first edition of Thomson and Tait's 'Natural Philosophy' (vol. 1, p. 38, § 58), it is shown that any two simple harmonic functions of one period can be compounded to a single simple harmonic function of the same period, and that the vector, representing the compounded function, is obtained from those representing the component functions by the ordinary process of vector addition.

This device has proved useful for many purposes, but it has proved especially valuable in connexion with alternate current problems. Its application to such cases was first clearly pointed out by Mr. T. H. Blakesley more than twelve years since. The use of it in alternate current work has gradually developed into what may be described as a vector, or network, method of representing alternate current quantities. In this method the length of the vector denotes the magnitude of the current or voltage, while the angle between any two vectors represents the time which elapses between the instants at which the maximum values of the corresponding quantities occur. In particular, if one line represents the voltage on a conductor, and a second line denotes the strength of current produced in it by this voltage, the power absorbed in the conductor is measured by the product of the length of the two lines into the cosine of the angle between them. It is to this possibility of

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deducing from the network figure the electrical *power* absorbed by the conductors, as well as the magnitude of each current, and voltage, that is chiefly due the usefulness and importance of this mode of representation.

It will be convenient here to describe briefly the mathematical basis on which the method rests. It will be evident that we may represent the value of a simple harmonic function $A \cos pt$ either by the projection of a revolving vector on a fixed line, or by the projection of a stationary vector on a revolving line. It will be more convenient for our purpose to adopt the latter plan. If, therefore, we have a vector α, of length A, the projection of this on a line supposed to revolve uniformly will always be equal to $A \cos pt$, provided the period of revolution T is such that $pT = 2\pi$, and provided also that the line coincides with the vector at the instant from which t is measured. We can, moreover, represent the value of $B \sin pt$ by the projection at the *same instant* on the *same revolving line* of a vector β of length B, provided only that the vectors α and β are perpendicular. It will also be evident that any other simple harmonic function of the same period

$$mA \cos pt + nB \sin pt$$

will be similarly represented by the projection of the vector

$$m\alpha + n\beta,$$

and the length of this vector will be equal to the maximum value of the function it represents.

Now, in any system of conductors traversed by alternate currents, if every electromotive force E_x is a simple harmonic function of one period, $E_x = V_x \sin (pt + \phi_x)$, and if the coefficients of induction and the capacities are all constant quantities, it follows mathematically that each current, and each voltage, can be expressed in the general form

$$F = \lambda \cos pt + \mu \sin pt,$$

which can thus be represented by the vector

$$v = \lambda\alpha + \mu\beta,$$

where α and β are the perpendicular vectors representing $\cos pt$ and $\sin pt$.

If α and β are unit vectors, the length of v is $\sqrt{(\lambda^2 + \mu^2)}$, which is the maximum value of the function F. The *magnitude* of an alternate current quantity is however estimated by the square root of its mean square, and not by its maximum value, and since

$$\frac{1}{T}\int_0^T F^2 dt = \tfrac{1}{2}(\lambda^2+\mu^2),$$

if we choose for α and β two perpendicular vectors of lengths numerically equal to $1/\sqrt{2}$, the length of $v = \lambda\alpha+\mu\beta$ will be equal to $\sqrt{\frac{1}{2}(\lambda^2+\mu^2)}$, or to the *magnitude* of F.

If now we consider the value of the mean product of two functions $F_1 \times F_2 =$

$$(\lambda_1 \cos pt + \mu_1 \sin pt)\,(\lambda_2 \cos pt + \mu_2 \sin pt),$$

we see that this mean product is

$$\tfrac{1}{2}(\lambda_1\lambda_2+\mu_1\mu_2),$$

or the same as the scalar product of the corresponding vectors,

$$(\lambda_1\alpha+\mu_1\beta)\,(\lambda_2\alpha+\mu_2\beta).$$

Every alternating current problem is reducible to the consideration of the current, voltage, and electrical power spent in the different conductors. Since all these quantities are varying periodically some average value has to be taken. For current and voltage the mode of averaging selected is always that obtained by taking the square root of the mean value of the square of the quantity during the period. The electrical power is, however, the mean value through the period of the product of the instantaneous values of voltage and current. Hence, if the current and voltage are represented by vectors in the manner described, the scalar product of the two vectors will represent the power. This vector method is particularly suited to alternate current problems, and has been very much used for such purposes by electricians, since it is simple to understand and to apply, and it can be used for the simple solution of many problems which would otherwise need the use of the calculus.

The method, however, assumes that all the voltages and currents considered vary according to a simple sine law. In ordinary commercial practice this is unfortunately not the case, owing chiefly to saturation and hysteresis phenomena in the magnetic circuits, and also to the fact that ordinary dynamos do not develop an electromotive force varying according to a simple harmonic law. The object of the present investigation is to show how far this method can be modified, and used correctly, in cases where the currents, although periodic, do not vary according to any simple law.

Since we shall have frequently to refer to the value of the square root of the mean square of a periodic function, we shall call such a quantity the *magnitude* of the periodic function. Thus, if v is a varying function of the time, periodically repeating itself after every interval T, the *magnitude* of v, which we shall denote by $\overline{\overline{v}}$, is given by the equation

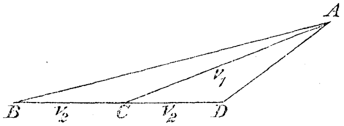
$$\overline{\overline{v}} = \textit{magnitude} \text{ of } v = \sqrt{\frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} v^2 dt}.$$

When we are considering two such functions, v_1 and v_2, the mean value of their product will be denoted by

$$\overline{v_1 v_2} = \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} v_1 v_2 dt.$$

The mode of measuring electrical power known as the Three Voltmeter Method,* is really dependent upon a curious property of periodic functions which may be stated as follows :—

If *any two arbitrary periodic functions*, v_1 and v_2, be represented in magnitude by the lengths of two sides of a triangle, and if the third side represents the magnitude of v (another periodic function equal to the sum, or difference, of the former two), then the mean value of the product of any two of these functions will be represented by the product of the lengths of the two corresponding sides and the cosine of the angle between them.



Thus, in the figure, if CA represents v_1, and BC (or CD) represents v_2, then BA will represent $v_1 + v_2$, and DA will represent $v_1 - v_2$; also the mean product of $(v_1 + v_2)v_2$ will be BA × BC cos ABC, the mean product $(v_1 - v_2)v_1$ will be DA × CA cos CAD, and so on. Thus we may regard v_1 and v_2 as vectors, the length of the vector representing the magnitude of the periodic function, and the scalar product of two vectors the mean product of the two corresponding functions. Any periodic function derivable from v_1 and v_2, by a linear relation, can be similarly represented in accordance with the ordinary properties of vectors, and the scalar product of any two such vectors will be equal to the mean product of the corresponding periodic functions.

In order to establish this we may proceed as follows :—

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* See "The Measurement of the Power given by any Electric Current to any Circuit," Professor W. E. Ayrton and Dr. W. E. Sumpner, 'Roy. Soc. Proc.,' April 9, 1891.

Whatever be the law of variation of v, v_1, and v_2, if

$$v = v_1 \pm v_2 \ldots\ldots\ldots\ldots\ldots\ldots (1),$$

then $$v^2 = {v_1}^2 \pm 2v_1v_2 + {v_2}^2$$

and, taking means, we have

$$\overline{v^2} = \overline{{v_1}^2} \pm 2\overline{v_1v_2} + \overline{{v_2}^2},$$

which equation is the same as

$$\left(\overset{=}{v}\right)^2 = \left(\overset{=}{v}_1\right)^2 \pm 2\overline{v_1v_2} + \left(\overset{=}{v}_2\right)^2 \ldots\ldots\ldots\ldots (2).$$

But if the lengths of the three sides of a triangle respectively represent $\overset{=}{v}$, $\overset{=}{v}_1$, and $\overset{=}{v}_2$, we have from geometry

$$\left(\overset{=}{v}\right)^2 = \left(\overset{=}{v}_1\right)^2 + \left(\overset{=}{v}_2\right)^2 \pm 2\left(\overset{=}{v}_1\right)\left(\overset{=}{v}_2\right)\cos\phi \ldots\ldots\ldots (3),$$

where ϕ is the angle between $\overset{=}{v}_1$ and $\overset{=}{v}_2$.

Comparing equations (2) and (3) with the figure, it will be seen that for the case represented by the line AB the positive sign must be taken in each equation, while for the other case, AD, the negative signs must be chosen. In either case we have $\overline{v_1v_2} = \left(\overset{=}{v}_1\right)\left(\overset{=}{v}_2\right)\cos\phi$. Moreover, if in equation (1) we transpose either v_1 or v_2, and square and proceed as before, we find that whichever two sides of the triangle ABC be chosen, the products of these sides into the cosine of the angle between them is equal to the mean product of the corresponding periodic functions.

The foregoing argument shows that *assuming it possible* to form a triangle with sides equal to the *magnitudes* of v, v_1, and v_2, such triangle has the properties stated. But we have yet to show that it is always possible to construct such a triangle, or, in other words, to show that any two of these quantities are together necessarily greater than the third.

Now if $$v = v_1 + v_2,$$

we find by squaring and taking the mean that

$$\left(\overset{=}{v}\right)^2 = \left(\overset{=}{v}_1\right)^2 + \left(\overset{=}{v}_2\right)^2 + 2\overline{v_1v_2};$$

we shall therefore prove that

$$\overset{=}{v} < \overset{=}{v}_1 + \overset{=}{v}_2,$$

provided we can establish the inequality

$$\left(\overset{=}{v}_1\right)\left(\overset{=}{v}_2\right) > \overline{v_1v_2} \ldots\ldots\ldots\ldots\ldots\ldots (4);$$

so that if we can show for *any* two periodic functions that the product of their magnitudes is necessarily greater than the mean value of their product, we shall have shown that it is always possible to construct a triangle the sides of which represent the magnitudes of any three periodic functions related as in equation (1).

The meaning of (4) can be illustrated by supposing that v_1 represents an alternating voltage applied to a conductor, and producing a current, v_2, through it. The right-hand element of (4) then represents the power absorbed, while the left-hand element is the product of volts and amperes. The fact that the power absorbed in an inductive circuit is always found to be less than the product of volts and amperes, is usually stated by asserting that the "power factor" is less than unity for such circuits. Experience has not shown any case in which the power factor of a circuit is greater than unity, but this of course is no evidence of the truth of (4), which must be proved for the general case in which the two periodic functions are perfectly arbitrary Fourier series.*

Let us therefore put

$$v_1 = \sum_m (A_{1m} \sin mpt + B_{1m} \cos mpt) \quad \dots\dots\dots\dots \quad (5).$$

$$v_2 = \sum_m (A_{2m} \sin mpt + B_{2m} \cos mpt) \quad \dots\dots\dots\dots \quad (6).$$

By taking the mean value of twice the square of each of these equations, we get

$$2\left(\overline{\overline{v_1}}\right)^2 = \sum_m (A_{1m}^2 + B_{1m}^2),$$

$$2\left(\overline{\overline{v_2}}\right)^2 = \sum_m (A_{2m}^2 + B_{2m}^2),$$

and by taking four times the value of the mean product we get

$$4\overline{v_1 v_2} = 2 \sum_m (A_{1m}A_{2m} + B_{1m}B_{2m}),$$

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* In the proof given, the two functions v_1 and v_2 are assumed to have fundamental periods that are commensurable with each other. This will always be so in practical cases, and what happens with functions of incommensurable periods is of but theoretical interest. It is, however, worth notice that no limitation is thereby imposed on the application of the theorems given in this paper. These theorems deal with mean products, and the truth of them is directly deducible from the fundamental inequality given in (4). Now when the periods are incommensurable the product v_1v_2 consists of terms such as $\sin mpt \sin npt$, where m and n are incommensurable, and as each of these can be expressed as the sum of two sinuous functions of angles denoted by $(m \pm n)pt$, the mean value is zero, the vectors representing v_1 and v_2 are perpendicular, and the truth of (4) is at once evident. But inasmuch as the value of $(m-n)$ may be infinitesimally small, it may be necessary to take an infinite time to determine the mean values.

and we shall have proved (4) if we can show that

$$4\left(\overline{\overline{v_1}}\right)^2\left(\overline{\overline{v_2}}\right) > \left(2\overline{v_1v_2}\right)^2,$$

or that

$$\left\{\sum_m\left(A_{1m}{}^2+B_{1m}{}^2\right)\right\}\times\left\{\sum_m\left(A_{2m}{}^2+B_{2m}{}^2\right)\right\}>\left\{\sum_m\left(A_{1m}A_{2m}+B_{1m}B_{2m}\right)\right\}^2.$$

This will be proved if we can establish the algebraic inequality—

$$\left(A_1^2+A_2^2+\ldots\ldots+A_n^2\right)\left(B_1^2+B_2^2+\ldots\ldots+B_n^2\right)$$
$$> \left(A_1B_1+A_2B_2+\ldots\ldots+A_nB_n\right)^2 \quad\ldots\ldots\quad (7).$$

Now it will be found quite easy to establish this inequality by an inductive method, but there is no need to do so here, since it is proved in the theory of determinants that

$$\begin{vmatrix} A_1^2+A_2^2+\ldots\ldots+A_n^2 & A_1B_1+A_2B_2+\ldots\ldots+A_nB_n \\ A_1B_1+A_2B_2+\ldots\ldots+A_nB_n & B_1^2+B_2^2+\ldots\ldots+B_n^2 \end{vmatrix}$$
$$= \Sigma\begin{vmatrix} A_1 & A_2 \\ B_1 & B_2 \end{vmatrix}^2,$$

and is necessarily positive.

It follows that the inequality (7) is true. In order that the inequality may become an equality the necessary and sufficient conditions are

$$A_1/B_1 = A_2/B_2 = \ldots\ldots = A_n/B_n.$$

In other words, the two Fourier series must bear a constant ratio to each other at every instant of time, and must be multiples of the *same* Fourier function. This is a limiting case which is not contemplated in (4), and with this exception only, we have shown* that :—

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* [The following much simpler proof has recently been noticed :—

Whatever v_1 and v_2 may be, we can always put

$$v_1 = a_1 + x, \qquad v_2 = a_2 + y,$$

where a_1 and a_2 are scalar constants given by the equations

$$a_1 = \overline{\overline{v_1}}, \qquad a_2 = \overline{\overline{v_2}}.$$

By squaring each expression and taking means we get

$$0 = 2a_1\overline{x} + \overline{x^2}, \qquad 0 = 2a_2\overline{y} + \overline{x^2} + \overline{y^2},$$

where by $\overline{x}$ is meant the arithmetical mean of x, &c.

Also, we obtain by multiplication and taking means,

$$\overline{v_1v_2} = a_1a_2 + \overline{xy} + a_1\overline{y} + a_2\overline{x}.$$

The product of the magnitudes of two periodic functions is always greater than the mean value of their product.

From this and from the foregoing reasoning, it follows that :—

Any two periodic functions can be represented by vectors in such a way that the length of each vector represents the magnitude of the function, and the scalar product of the vectors, the mean product of the functions. Any other function derived from the first two by means of a linear relation can be represented in magnitude by a vector derived by means of the same linear relation from the two original vectors. The scalar product of any two such vectors will be equal to the mean product of the corresponding functions.

Now suppose α and β are two vectors representing two arbitrary Fourier series, F_α and F_β. It follows from the law of vectors that any vector in the plane $\alpha\beta$ will be a linear function of α and β, and must therefore represent a Fourier series derivable from F_α and F_β in a linear manner. If, therefore, we have a third arbitrary Fourier series, F_γ, which cannot be expressed as a linear function of F_α and F_β, it follows that this cannot be represented by a vector in the same plane as α and β. It remains to show that a vector, γ, out of this plane can still be found to represent F_γ.

Let us first of all assume that three vectors α, β, γ (not in one plane) have been found to represent three Fourier series F_α, F_β, and F_γ. We can then show that any linear function of α, β, γ, similarly represents the Fourier series, which is the same linear function of F_α, F_β, F_γ.

For if, in accordance with the notation already used, we denote by the symbols

$\overline{\overline{F_\alpha}}$	the square root of the mean square of the value of F_α, *i.e.*, the *magnitude* of F_α,
$\overline{F_\beta F_\gamma}$	the mean value of the product $F_\beta F_\gamma$,
$\overline{\overline{\alpha}}$	the length of the vector α,
$\overline{\beta\gamma}$	the scalar product of the vectors β, γ,

If we substitute from the above relations, we get

$$\overline{v_1v_2} = a_1a_2 + \overline{xy} - \tfrac{1}{2}\left\{\frac{a_1}{a_2}\overline{y^2} + \frac{a_2}{a_1}\overline{x^2}\right\} = a_1a_2 - \frac{\overline{Z^2}}{2a_1a_2},$$

where Z represents the value of $a_2x - a_1y$.

It follows that $\overline{v_1v_2}$ is always less than a_1a_2, except in the limiting case, in which Z is zero at all parts of the period. This will only occur if

$$a_2x = a_1y$$

at every instant, or when

$$a_2v_1 = a_1v_2.$$

—June 11, 1897.]

we then have $\overset{=}{\alpha} = \overline{\overline{F}}_\alpha, \qquad \overset{=}{\beta} = \overline{\overline{F}}_\beta, \qquad \overset{=}{\gamma} = \overline{\overline{F}}_\gamma,$

$$\overline{\beta\gamma} = \overline{F_\beta F_\gamma}, \quad \overline{\gamma\alpha} = \overline{F_\gamma F_\alpha}, \quad \overline{\alpha\beta} = \overline{F_\alpha F_\beta}.$$

If, now, we compare the value of the square of any vector $\lambda\alpha + \mu\beta + \nu\gamma$ with the value of the mean square of the corresponding Fourier series $\lambda F_\alpha + \mu F_\beta + \nu F_\gamma$, we find, by expanding and using the above relations, that the expressions are identical; and, moreover, it is clear that the scalar product of any two such vectors

$$(\lambda_1\alpha + \mu_1\beta + \nu_1\gamma)\ (\lambda_2\alpha + \mu_2\beta + \nu_2\gamma)$$

is identical with the mean value of the product of the corresponding pair of Fourier series

$$(\lambda_1 F_\alpha + \mu_1 F_\beta + \nu_1 F_\gamma)\ (\lambda_2 F_\alpha + \mu_2 F_\beta + \nu_2 F_\gamma).$$

We have now, however, to show that whatever F_α, F_β, F_γ may be, it is always possible to find three vectors, α, β, γ, to represent them.

For this purpose let us consider two other Fourier functions, F_u and F_v, derived in a linear manner from the given functions, F_α, F_β, F_γ, in accordance with the equations

$$F_u = \begin{vmatrix} F_\alpha & F_\beta \\ a & b \end{vmatrix} \quad \text{and} \quad F_v = \begin{vmatrix} F_\alpha & F_\gamma \\ a & c, \end{vmatrix}$$

where a, b, and c are scalars determined from the mean products,

$$\begin{aligned} a &= \overline{F_\alpha F_\alpha}, \\ b &= \overline{F_\alpha F_\beta}, \\ c &= \overline{F_\alpha F_\gamma}. \end{aligned}$$

It will then be seen that

$$\overline{F_\alpha F_u} = 0 \quad \text{and} \quad \overline{F_\alpha F_v} = 0,$$

and whatever F_u and F_v may be there is a real angle, ϕ, for which

$$\overline{F_u F_v} = \overline{\overline{F}}_u \times \overline{\overline{F}}_v \times \cos\phi,$$

since the product $\overline{F_u F_v}$ must be less than $\overline{\overline{F}}_u \times \overline{\overline{F}}_v$.

If, therefore, we take two planes inclined to each other at an angle, ϕ, and take two vectors, u, v, in these planes, drawn perpendicular to the line of intersection, and of such length that

$$\overset{=}{u} = \overline{\overline{F}}_u \quad \text{and} \quad \overset{=}{v} = \overline{\overline{F}}_v,$$

and if, in addition, we choose a vector, α, along the line of intersection of these planes, and of such length that

$$\overset{=}{\alpha} = \overline{\overline{F}}_\alpha,$$

we shall then have

$$\overline{\alpha u} = 0 = \overline{F_\alpha F_u}$$

$$\overline{\alpha v} = 0 = \overline{F_\alpha F_v}$$

$$\overline{uv} = \overline{\overline{u}} \times \overline{\overline{v}} \times \cos\phi = \overline{\overline{F_u}} \times \overline{\overline{F_v}} \times \cos\phi = \overline{F_u F_v},$$

or the three vectors α, u, v completely represent the series F_α, F_u, F_v.

Since β and γ are linear functions of α, u, and v, while F_β, F_γ are the same linear functions of F_α, F_u, and F_v, it follows from the foregoing reasoning that β, γ also represent the periodic functions F_β, F_γ.

It is thus established that:—

Any three periodic functions can be represented by vectors in such a way that the length of the vector represents the magnitude of the function, while the scalar product of any two vectors represents the mean value of the product of the corresponding functions. Any other periodic function derivable from the original three, in a linear manner, can be similarly represented by a vector, derivable from the original three vectors by means of the same linear relation.

It will, however, be seen that it is impossible, in three-dimensioned space, to find vectors to represent four arbitrary periodic functions, unless a linear relation connects them. For instance, the functions $\sin pt$, $\cos pt$, $\sin 2pt$, $\cos 2pt$ cannot be so represented. The mean product of any two of these is zero, and it is impossible in three-dimensioned space to find four lines each of which is at right angles to the other three.

In discussing problems the solutions of which are expressible in terms of the mean square of a Fourier function, or the mean product of two such functions, it may sometimes prove advantageous to make use of the foregoing theorem. As a rule the complicated Fourier series occurring in actual problems are obtained in an indirect and somewhat artificial manner by expressing in precise mathematical language the value of some periodic quantity which can be represented graphically in a very simple way. In some cases the curve, though perfectly determinate, follows no recognised mathematical law; in other cases it consists of a series of discontinuous curves, each of simple mathematical character. In all such instances the mean square of the ordinate of such a curve, and the mean product of the corresponding ordinates of two such curves, can more easily and quickly be obtained than the Fourier expansions themselves. If the problem only demands a knowledge of these mean squares and mean products, the vector method will probably be the simplest one to apply, and possibly many problems exist which may be solved in this manner. This seems the more likely when it is remembered that, in problems involving periodic forces and fluxes, the *energy* depends upon the *mean product of the force and its corresponding flux*, or, if the forces and fluxes are proportional, upon the *mean square* of the value of either. The theorem, however, seems most immediately applicable to the case of alternate current problems.

To apply the theorem to alternating currents, let us first of all consider the case of a network of conductors in which there are only three arbitrary periodic electromotive forces, E_1, E_2, and E_3, acting in the branches. If the resistances are all non-inductive, it will always then be possible to express the current in each branch, and the potential between any two points, as a linear function of E_1, E_2, and E_3, the coefficients only involving the resistances of the network branches. If α_1, α_2, and α_3 are vectors representing the periodic functions E_1, E_2, E_3, it will be possible to represent all the potentials and all the currents involved by means of a network of vectors.

A case in which the resistance of one of the branches is inductive, owing to its possession of a constant coefficient of self-induction, can best be treated by assuming that such a branch possesses an arbitrary electromotive force E, related to the current C, passing through the branch in a manner given by the equation

$$E = L\frac{dC}{dt}.$$

We know nothing of the relation between the vectors representing E and C, except that their scalar product is zero, and they are in consequence perpendicular to each other. No energy is wasted, due to self-induction, since

$$\int_0^T CL\frac{dC}{dt}\,dt = 0.$$

This relation gives a certain connexion between the coefficients which express C as a linear function of the vectors denoting the electromotive forces, but does not make it possible to express E as a linear function of the other electromotive forces E_1 and E_2. For although we may put $E = \lambda_1 E_1 + \lambda_2 E_2$, the coefficients λ_1 and λ_2 will not be scalar quantities, since they will involve not only the resistances of the branches of the network, but also the operator d/dt. It will always be possible to find a vector to represent E, but only in special cases will this vector be in the same plane as the vectors denoting E_1 and E_2.

Similarly a case in which two points of the network are connected with the terminals of a condenser of capacity K can best be treated by assuming that the condenser is replaced by a conductor possessing an arbitrary electromotive force E, related to the current passing through the branch in a manner given by the equation

$$\mathrm{E} = -\frac{1}{\mathrm{K}}\int_0^t \mathrm{C}dt.$$

The vectors representing E and C are such that their scalar product is zero, and the same remarks apply as in the preceding case.

The case of a transformer can be similarly treated. The reaction due to the varying induction of the core produced by a current C through the primary, must be treated as an arbitrary electromotive force E. The case will differ from that of constant self-induction, in that the vectors representing E and C are at right angles in the latter case, but not in the former, since, owing to hysteresis and eddy currents, the core absorbs energy, and the scalar product of the vectors E and C cannot be zero.

It will thus be seen that *the effect of inductance in a branch of the network is the same as that of an arbitrary electromotive force placed there, whether the inductance be due to constant self-induction, to condenser action, or to transformer action.* With this understanding it follows that *a vector figure can be drawn in space, completely representing the alternating currents and potentials in a network of conductors, provided the number of independent alternating electromotive forces does not exceed three. This figure can be drawn in a plane if the number of such electromotive forces is not more than two.* Of course, where a linear relation, with scalar coefficients, connects three electromotive forces, only two of these are to be regarded as independent.

Now the vector method which has hitherto been so largely used in explaining alternating current phenomena, will not be really serviceable in the new and extended application here suggested, except in cases in which the vector figure is either confined to one plane, or can, with all essential accuracy, be treated as if that were the case.

Fortunately, several circumstances conspire to simplify the application of the theorem to alternate current problems. In the first place the vectors which are needed are divisible naturally into two sets: one representing the currents, and the other the voltages. It is never necessary to find the product of two of these vectors unless they belong to different sets. Although in some cases it may not be possible to construct the complete vector figure in one plane, it may still be possible to construct two figures—one for each set. In practical cases one of these figures (say the current vectors) will almost certainly be confined to one plane, and if the vectors of the other (or voltage) network be projected perpendicularly on to this plane, the product of any projected voltage vector and the appropriate current vector will accurately represent the corresponding quantity of power, since it is a well-known property of vectors that the scalar product of any two vectors is the same as the scalar product of one of them and the perpendicular projection of the other on any plane containing the first.

It will only rarely happen that a plane figure will fail to give a sufficiently accurate representation of the relations between the volts and amperes in alternate current circuits. These circuits are generally simple in character. Complication is not caused by a multitude of conductors unless some of these are inductive. A network of non-inductive resistances may involve a large number of currents and potentials, but they are all connected by linear relations, and the construction of the vector network presents no difficulty. Wherever a transformer is used, the primary and secondary circuits can be treated as quite distinct. The vector figures representing them can indeed be drawn to different scales, the only necessary connexion between them being the relation which connects the power put into the primary core of the transformer with that taken from the secondary. As a type of the most complicated circuit likely to occur, we may take the case of an alternator feeding the primary coil of a transformer, the primary terminals of which are also connected through a condenser. Here we have three inductive circuits in parallel. There are only three currents to consider. One of these can be expressed as the sum of the other two, and hence the current vectors lie in a plane. There is only one voltage. This may not lie in the plane of the currents, but its projection on this plane will give all that is needed. Let E be the electromotive force of the dynamo, R the resistance of the armature and leads up to the terminals of the transformer, kept at voltage V, and let C be the armature current, we then have

$$E = RC + V.$$

The vectors representing E and V may not be in the plane of the currents, but their projections on this plane are vectors which are connected with each other by the same linear relation as in the above equation, and in nearly all practical cases the projections may be taken to represent the voltages.

The chief purpose for which this vector method has hitherto been used is to explain or to predict the nature of the changes caused in alternate current circuits by variations in resistance, self-induction, or capacity, and to deduce the relations which must be fulfilled in order that certain results may be obtained. It has been possible to do this because from the network representing one particular case it has been easy to draw the figures representing other cases by merely increasing the length of certain of the lines in the proper proportion.

It will be evident that the same process is equally applicable when the currents are no longer of a simple sinuous character, provided only that the network representing one particular case can be properly drawn; for, however complicated the network may become when it represents several such cases, it is obtained from the original network by increasing some of the vectors to correspond with the changes in the conductors. If we have a current, C, passing through (1) a wire of resistance R, (2) a coil of self-induction L, or (3) a condenser of capacity K, the voltages at the ends of these conductors are given respectively by the expressions

$$\mathrm{RC}, \quad \mathrm{L}\frac{d\mathrm{C}}{dt}, \quad \text{and} \quad -\frac{1}{\mathrm{K}}\int_0^t \mathrm{C}\,dt,$$

and hence the vectors representing them are proportional respectively to

$$\mathrm{R}, \ \mathrm{L}, \text{ and } 1/\mathrm{K},$$

and the effect of an alteration in R, L, or K can generally be easily indicated on the network figure.

It may, therefore, be taken as established that the vector method of representing alternate current quantities is as applicable when the currents and potentials vary in a manner widely different from the sine law as when they are simply sinuous. But it is desirable in this connexion to repeat and emphasise an old warning about the continued use of coefficients of self and mutual induction in the theoretical investigations of alternate current problems. These coefficients may be of service in connexion with coils not possessing iron cores, but they are very misleading in commercial cases in which iron is present. By no choice of the value of these coefficients is it possible to represent the phenomena of hysteresis. The presence of the latter implies that the current has a different law of variation from that of the voltage, while the assumption of constant induction coefficients necessitates that if either the current or potential follows a sine law the other must also. Again, constant self-induction involves the assumption that there is no loss of energy in the magnetic changes going on in the iron, an assumption which is not only known to be wrong, but one which involves an error of serious practical importance. It has already* been pointed out that the difficulty of reconciling theoretical formulæ with experimental results does not arise so much from assuming that the currents vary according to a simple sine law, as from the quite unjustifiable assumption and use of constant coefficients of induction.

* See 'Electrician,' June 5 1891, "Alternate Current Theory," by Dr. W. E. Sumpner.